ELASTOPLASTIC PROBLEM FOR A THIN PLATE WEAKENED BY A PERIODIC SYSTEM OF ROUND APERTURES

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## INTRODUCTION

The problem of elasticity theory for a plate weakened by a periodic system of apertures has been treated in a series of papers [1, 2]. As stresses increase, plastic zones arise in the plate near the apertures. The position of the plastic regions has a periodic character. The elastoplastic problem for a thin plate with a single aperture was solved in [3]. A series of papers has been devoted to periodic elastoplastic problems for a thin plate [4, 5]. A method of approximating the stress function in the plastic region by a biharmonic function was used in [4] to solve the problem. By contrast with [4, 5], where the method of perturbations was used, the present paper applies another method for solving the elastoplastic problem which allows a solution to be obtained for any relative dimensions of the region.

We shall consider a plate with identical round apertures of radius $R (R < 1)$ centered at the points

$$P_m = m\omega, \ (m = 0, \ \pm 1, \ \pm 2, \ldots), \ \omega = 2.$$

We shall denote the contour of the aperture with center at the point $P_m$ by $L_m$, the corresponding elastoplastic boundary by $\Gamma_m$, and the exterior of the contours $\Gamma_m$ by $D_z$. Let a constant normal load $\sigma_r = p$ be applied to the contour of aperture $L_m$, and let the tangential component be equal to zero: $\tau_{r\theta} = 0$ ($r$, $\theta$ are polar coordinates). Let the constant mean stresses exist in the plate, $\sigma_x = \sigma_x^\infty$, $\sigma_y = \sigma_y^\infty$, $\tau_{xy} = 0$ (these are tensions at infinity).

By way of the plasticity condition we adopt the Tresca–St. Venant condition and assume that the inequality $\sigma_\theta \geq \sigma_r > 0$ is satisfied in the plastic region. The characteristics in the plastic zone are radial straight lines, and the stresses are equal to [6]

$$\sigma_r = \sigma_s + (p - \sigma_s) R/r, \ \sigma_\theta = \sigma_s, \ \tau_{r\theta} = 0. \tag{1}$$

Here $\sigma_s$ is the yield stress of the material for simple stress. For the inequality $\sigma_\theta \geq \sigma_r > 0$ to be satisfied the load should clearly satisfy the condition $p \leq \sigma_s$.

In the elastic region the stresses are determined from the Kolosov–Muskhelishvili formulas [7]

$$\begin{gathered} \sigma_r + \sigma_\theta = 4 \operatorname{Re} \Phi(z), \\ \sigma_\theta - \sigma_r + 2i\tau_{r\theta} = 2[\bar{z}\Phi'(z) + \Psi(z)] \, e^{2i\theta}. \end{gathered} \tag{2}$$

All the stresses are continuous on the unknown contour $\Gamma_m$ dividing the elastic and plastic regions. Using Eqs. (1), (2) we obtain the following conditions on the contour $\Gamma_m$:

$$\begin{gathered} 4 \operatorname{Re} \Phi(z) = 2\sigma_s + R(p - \sigma_s)/r; \\ \bar{z}\Phi'(z) + \Psi(z) = R(\sigma_s - p)/2r] e^{-2i\theta}. \end{gathered}$$

We now pass to the parametric $\zeta$ plane with the help of the transformation $z = \omega(\zeta)$. The analytic function $z = \omega(\zeta)$ performs the conformal mapping of the region $D_z$ onto the region $D_\zeta$ in the $\zeta$ plane, which is the exterior of the circles $l_m$ of radius $\lambda$ and centers at the points $P_m$.

We obtain the following boundary-value problem on $l_m$ for determining the three analytic functions $\varphi(\zeta) = \Phi[\omega(\zeta)]$, $\psi(\zeta) = \Psi[\omega(\zeta)]$, and $\omega(\zeta)$:

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Lipetsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 174-179, September-October, 1976. Original article submitted November 20, 1975.

$$4\,\mathrm{Re}\,\varphi(\zeta) = 2\sigma_s + R(p - \sigma_s)/\sqrt{\omega(\zeta)\overline{\omega(\zeta)}}; \tag{3}$$

$$\overline{\omega(\zeta)}/\omega'(\zeta)\varphi'(\zeta) + \psi(\zeta) = R(\sigma_s - p)\overline{\omega(\zeta)}/2\omega(\zeta)\sqrt{\omega(\zeta)\overline{\omega(\zeta)}}. \tag{4}$$

We shall look for the required functions in the form of the series

$$\varphi(\zeta) = \frac{\sigma_x^\infty + \sigma_y^\infty}{4} + \alpha_0 + \sum_{k=0}^{\infty} \alpha_{2k+2} \frac{\lambda^{2k+2}\rho^{(2k)}(\zeta)}{(2k+1)!}; \tag{5}$$

$$\psi(\zeta) = \frac{\sigma_y^\infty - \sigma_x^\infty}{2} + \sum_{k=0}^{\infty} \beta_{2k+2} \frac{\lambda^{2k+2}\rho^{(2k)}(\zeta)}{(2k+1)!} - \sum_{k=0}^{\infty} \alpha_{2k+2} \frac{\lambda^{2k+2}s^{(2k+1)}(\zeta)}{(2k+1)!}; \tag{6}$$

$$\omega(\zeta) = \zeta + \sum_{k=0}^{\infty} A_{2k+2} \frac{\lambda^{2k+2}\rho^{(2k-1)}(\zeta)}{(2k+1)!}, \tag{7}$$

where

$$\rho(\zeta) = \left(\frac{\pi}{\omega}\right)^2 \frac{1}{\sin^2\left(\frac{\pi}{\omega}\zeta\right)} - \frac{1}{3}\left(\frac{\pi}{\omega}\right)^2;$$

$$s(\zeta) = \sum_m{}' \left[\frac{P_m}{(\zeta - P_m)^2} - \frac{2\zeta}{P_m^2} - \frac{1}{P_m}\right] (m = 0, \pm 1, \pm 2, \ldots).$$

The prime on the sum denotes that the index m = 0 is excluded from the summation. We now give the relations which the coefficients of Eqs. (5)-(7) must satisfy. From the conditions for symmetry relative to the coordinate axes we find that

$$\mathrm{Im}\,\alpha_{2k+2} = \mathrm{Im}\,\beta_{2k+2} = \mathrm{Im}\,A_{2k+2} = 0,\ k = 0, 1, 2, \ldots.$$

It follows that

$$\alpha_0 = (\pi^2/24)\beta_2\lambda^2$$

from the condition that the principal vector of the forces acting on an arc joining two congruent points in $D\xi$ should be zero. Since the periodicity conditions are satisfied, the system of boundary conditions (3), (4) on $l_m$ (m = 0, ±1, ±2, ...) is replaced by two functional equations, for example, on the contour $l_0$.

In order to construct the equations for the remaining coefficients of the functions $\varphi(\zeta)$, $\psi(\zeta)$, and $\omega(\zeta)$, we expand these functions in Laurent series in the neighborhood of the point $\zeta = 0$:

$$\varphi(\zeta) = \frac{1}{4}(\sigma_x^\infty + \sigma_y^\infty) + \alpha_0 + \sum_{k=0}^{\infty} \alpha_{2k+2}\frac{\lambda^{2k+2}}{\zeta^{2k+2}} + \sum_{k=0}^{\infty} \alpha_{2k+2}\lambda^{2k+2} \sum_{j=0}^{\infty} r_{j,k}\zeta^{2j}; \tag{8}$$

$$\psi(\zeta) = \frac{1}{2}(\sigma_y^\infty - \sigma_x^\infty) + \sum_{k=0}^{\infty} \beta_{2k+2}\frac{\lambda^{2k+2}}{\zeta^{2k+2}} + \sum_{k=0}^{\infty} \beta_{2k+2}\lambda^{2k+2} \sum_{j=0}^{\infty} r_{j,k}\zeta^{2j} -$$

$$- \sum_{k=0}^{\infty} (2k+2)\alpha_{2k+2}\lambda^{2k+2} \sum_{j=0}^{\infty} (2j + 2k + 2) r_{j,k}\zeta^{2j}; \tag{9}$$

$$\omega(\zeta) = \zeta - \sum_{k=0}^{\infty} A_{2k+2}\frac{\lambda^{2k+2}}{(2k+1)}\frac{1}{\zeta^{2k+1}} + \sum_{k=0}^{\infty} A_{2k+2}\lambda^{2k+2} \sum_{j=0}^{\infty} \frac{r_{j,k}\zeta^{2j+1}}{(2j+1)},$$

$$r_{j,k} = \frac{(2j+2k+1)!\,g_{j+k+1}}{(2j)!\,(2k+1)!\,2^{2j+2k+2}},\quad g_{j+k+1} = 2\sum_{m=1}^{\infty}\frac{1}{m^{2j+2k+2}}. \tag{10}$$

Expanding the right-hand sides of Eqs. (3), (4) in Laurent series, substituting the expansions Eqs. (8)-(10) into the boundary conditions (3), (4) on the contour $l_0(\zeta = \lambda e^{i\theta})$ in place of $\varphi(\zeta)$, $\psi(\zeta)$, and $\omega(\zeta)$, and comparing the coefficients of $e^{2ik\theta}$ (k = 0, ±1, ±2, ...), we obtain an infinite system of nonlinear algebraic equations in $\alpha_{2k}$, $\beta_{2k}$, $A_{2k}$. The equations for the first approximation are given below:

$$aX - a_1Y - A_2Z = B\left[kb + \frac{1}{2}(k_1 + k_2)b_1\right],$$

$$aY - A_2X = B\left(\frac{1}{2}b_1k + k_1b\right),$$

$$aZ + a_1X = B\left(k_2b + \frac{1}{2}b_1k\right),\ 2\alpha_2(1 + \lambda^4 r_{1,0}) = -Bb_1,$$

$$\frac{1}{2}(\sigma_x^\infty + \sigma_y^\infty) - \sigma_s + 2(\alpha_0 + \alpha_2\lambda^2 r_{0,0}) = -Bb,$$

$$\lambda^2\left[d\left(b^2 + \frac{1}{2}b_1^2\right) + 2d_1bb_1\right] = 1,\ dbb_1 + d_1\left(b^2 + \frac{1}{2}b_1^2\right) = 0,$$

$$X = 2\alpha_2A_2 + \frac{2}{3}\alpha_2A_2\lambda^8 r_{1,0}^2 + a\beta_2 + A_2\beta_4\lambda^4 r_{1,0} + A_2\gamma_0,$$

$$Y = -2\alpha_2a + a\beta_4 + A_2\beta_2,\ a = 1 + A_2\lambda^2 r_{0,0},$$

$$Z = 2a\alpha_2\lambda^4 r_{1,0} + a\gamma_0 + A_2\gamma_1 + A_2\beta_2\lambda^4 r_{1,0},$$

$$a_1 = \frac{1}{3}A_2\lambda^4 r_{1,0},\ B = \frac{1}{2}R(\sigma_s - p),$$

$$k = a^2 - A_2^2 + \frac{1}{3}A_2^2\lambda^8 r_{1,0}^2,\ k_1 = aA_2\left(1 + \frac{1}{3}\lambda^4 r_{1,0}\right),$$

$$k_2 = aA_2(\lambda^4 r_{1,0} - 1),\ d = a^2 + A_2^2\left(1 + \frac{1}{9}r_{1,0}^2\lambda^8\right),$$

$$d_1 = -aA_2\left(1 - \frac{1}{3}\lambda^4 r_{1,0}\right),$$

$$\gamma_0 = \frac{1}{2}(\sigma_y^\infty - \sigma_x^\infty) + \beta_2\lambda^2 r_{0,0} + \beta_4\lambda^4 r_{0,1} - 4\alpha_2\lambda^2 r_{0,0},$$

$$\gamma_1 = \beta_2\lambda^4 r_{1,0} + \beta_4\lambda^6 r_{1,1} - 8\alpha_2\lambda^4 r_{1,0}.$$

Results of the calculations in the first two approximations are given in Table 1 for $\sigma_x^\infty = \sigma_y^\infty = q$, where $a_* = 2B$. The parameter $\lambda$ is given in Fig. 1 as a function of the magnitude of the applied load $q/\sigma_s$ for $p = 0$ and for some values of the aperture radius $R = 0.5$; 0.4; 0.3; 0.2; 0.1 (curves 1–5).

Setting $\zeta = \lambda e^{i\theta}$, in Eq. (10), we obtain the equation for the elastoplastic boundary:

$$r = |\omega(\lambda\, e^{i\theta})| = f(\theta).$$

In the first approximation

$$r^2 = \lambda^2(d + 2d_1\cos 2\theta).$$

In this case

$$r_{\max} = \lambda\left[1 + A_2\left(-1 + \lambda^2\sum_{j=0}^{\infty}\frac{r_{j,0}}{2j+1}\lambda^{2j}\right)\right];$$
$$r_{\min} = \lambda\left[1 + A_2\left(1 + \lambda^2\sum_{j=0}^{\infty}\frac{(-1)^j r_{j,0}}{2j+1}\lambda^{2j}\right)\right]. \qquad (11)$$

The elastoplastic boundary is represented in Fig. 2 for the case $R = 0.3$, $p = 0$, $q/\sigma_s = 0.627$ ($\lambda = 0.7$, $r_{\max} = 0.85$, $r_{\min} = 0.419$).

The condition $r_{\min} \geq R$ determines the least load for which the contour of the aperture is completely surrounded by the plastic zone. For $r_{\max} \leq 1$, Eq. (11) allows us to find the largest load for which the plastic zones touch each other. Until now it has been assumed that the load p satisfies the inequality $0 \leq p \leq \sigma_s$.

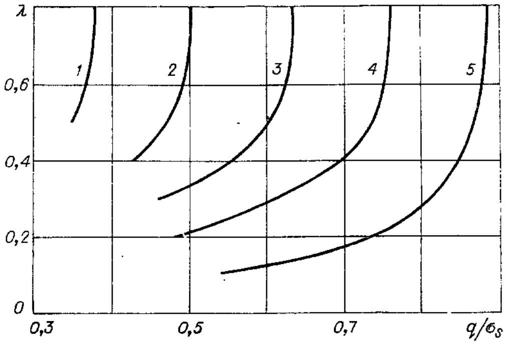


Fig. 1

TABLE 1

| $\lambda$ | 0,2 | 0,3 | 0,4 | 0,5 | 0,6 | 0,7 | 0,8 |
|---|---|---|---|---|---|---|---|
| | | | | First approximation | | | |
| $\beta_2/\alpha_*$ | 2,50727 | 1,68811 | 1,29244 | 1,06681 | 0,92563 | 0,83308 | 0,77382 |
| $\beta_4/\alpha_*$ | —0,07198 | —0,09189 | —0,10138 | —0,10706 | —0,11571 | —0,13014 | —0,14761 |
| $A_2$ | —0,05883 | —0,11561 | —0,17187 | —0,22093 | —0,25989 | —0,28991 | —0,31452 |
| $\alpha_2/a$ | 0,07513 | 0,10157 | 0,11876 | 0,12818 | 0,13271 | 0,13273 | 0,14011 |
| $b$ | 5,01421 | 3,37443 | 2,58053 | 2,12735 | 1,85017 | 1,67710 | 1,57037 |
| $b_1$ | —0,30073 | —0,40774 | —0,48074 | —0,52512 | —0,56103 | —0,60252 | —0,62014 |
| | | | | Second approximation | | | |
| $\beta_2/a_*$ | 2,50727 | 1,68814 | 1,29249 | 1,06696 | 0,92584 | 0,83377 | 0,77478 |
| $\beta_4/a_*$ | —0,07198 | —0,09197 | —0,10147 | —0,10755 | —0,11686 | —0,13224 | —0,15362 |
| $\beta_6/a_*$ | 0,01456 | 0,03887 | 0,06877 | 0,09723 | 0,12270 | 0,15036 | 0,18855 |
| $A_2$ | —0,05883 | —0,11568 | —0,17291 | —0,22288 | — 0,26456 | —0,29878 | —0,32379 |
| $A_4$ | —0,00576 | —0,02163 | —0,04516 | —0,06745 | —0,08735 | —0,11922 | —0,18572 |
| $\alpha_2/a_*$ | 0,07513 | 0,10161 | 0,11893 | 0,12873 | 0,13418 | 0,13853 | 0,14297 |
| $\alpha_4/a_*$ | —0,00089 | —0,00268 | —0,00572 | —0,01013 | —0,01548 | —0,02051 | —0,02361 |
| $b$ | 5,01421 | 3,37446 | 2,58064 | 2,12881 | 1,85116 | 1,67844 | 1,57724 |
| $b_1$ | —0,30073 | —0,40783 | —0,48082 | —0,52827 | —0,56507 | —0,60706 | —0,66258 |
| $b_2$ | 0,00356 | 0,01066 | 0,02256 | 0,03929 | 0,05821 | 0,07292 | 0,07486 |

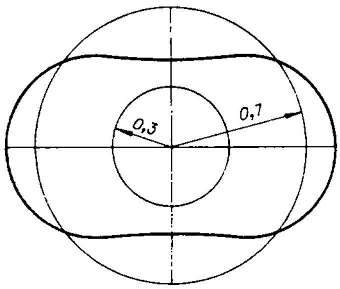


Fig. 2

Now let the load p vary within the limits $0 \geq p \geq -\sigma_s$. In this case, provided that the Tresca–St. Venant flow condition is satisfied, the stresses in the plastic zone are determined by the following formulas [6] for $R \leq r \leq R\exp(-p/\sigma_s)$,

$$\sigma_r = p + \sigma_s \ln(r/R), \quad \tau_{r\theta} = 0,$$
$$\sigma_\theta = p + \sigma_s + \sigma_s \ln(r/R), \quad \sigma_\theta - \sigma_r = \sigma_s;$$

for $r \geqslant R\exp(-p/\sigma_s)$,

$$\sigma_r = \sigma_s - (\sigma_s/r)R\exp(-p/\sigma_s),$$
$$\sigma_\theta = \sigma_s, \quad \tau_{r\theta} = 0.$$

We note that in this case all the solutions of the elastoplastic problem obtained previously will be valid only on condition that the elastoplastic boundary completely surrounds the circle of radius $R\exp(-p/\sigma_s)$. It then suffices to make the following formal substitutions everywhere in the solutions: p is replaced by zero and R, by $R\exp(-p/\sigma_s)$.

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